On Groups with Automorphisms Whose Fixed Points are Engel

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(This work was partially supported by the FAPDF, Brazil)

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ISCHIA GROUP THEORY 2018

Abstract

The purpose of this poster is to present new achievements on following problem.

Problem: Let A be a group of automorphisms of a group G. What is the influence of the structure of centralizers $C_G(a)$, where $a \in A$, has over the structure of G?

Let q be a prime and A a non-cycle elementary abelian q-group acting coprimely on a finite or profinite group G. We show that if for all $a \in A^{\#}$ the centralizers $C_G(a)$ satisfy some natural Engel conditions, then the whole group G satisfies similar conditions.

Keywords: Finite Groups, Profinite Groups, Automorphisms, Centralizers, Engel Elements

Introduction

Let A be a finite group acting on a finite group G. Many well-known results show that the structure of the centralizer of A

$$C_G(A) = \{x \in G | x^a = x \text{ for any } a \in A\}$$

has influence over the structure of G. For example:

Theorem 1 (J. G. Thompson – 1959) *Let A be a finite group of prime order acting on a finite group G. If $C_G(A) = 1$, then G is nilpotent.*

Theorem 2 (G. Higman – 1957) *If G is a nilpotent group admitting an automorphism φ of prime order q and such that $C_G(\varphi) = 1$, then the nilpotency class of G is q-bounded.*

The influence is especially strong if $(|A|, |G|) = 1$, that is, the action of A on G is coprime. Following the solution of the restricted Burnside problem it was discovered that the exponent of $C_G(A)$ may have strong impact over the exponent of G. The following result was proved in [7].

Theorem 3 (E.I. Khukhro, P. Shumyatsky – 1999) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ has exponent dividing n for each $a \in A^{\#}$. Then the exponent of G is $\{n, q\}$-bounded.*

The proof of the above result involves a number of deep ideas. In particular, Zelmanov's techniques that led to the solution of the restricted Burnside problem are combined with the Lubotzky–Mann theory of powerful p-groups, and a theorem of Bahturin and Zaicev on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI.
A profinite (non-quantitative) version of the above theorem was established in [8].

Theorem 4 (P. Shumyatsky – 2002) *Let q be a prime and A an elementary abelian q-group of order at least q^2. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is torsion for each $a \in A^{\#}$. Then G is locally finite.*

In [9] the situation where the centralizers $C_G(a)$ consist of Engel elements was dealt with. If x, y are elements of a (possibly infinite) group G, the commutators $[x,_n y]$ are defined inductively by the rule

$$[x,_0 y] = x, \quad [x,_n y] = [[x,_{n-1} y], y] \quad \text{for all } n \geq 1.$$

An element x is called a (left) Engel element if for any $g \in G$ there exists n, depending on x and g, such that $[g,_n x] = 1$. A group G is called Engel if all elements of G are Engel. The element x is called a (left) n-Engel element if for any $g \in G$ we have $[g,_n x] = 1$. The group G is n-Engel if all elements of G are n-Engel.

Main Results

The following result was proved in [9].

Theorem 5 (P. Shumyatsky, D.S.S – 2016) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^{\#}$ every element of $C_G(a)$ is n-Engel in G. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.*

A profinite (non-quantitative) version of the above theorem was established in the recent work [4].

Theorem 6 (C. Acciarri, P. Shumyatsky, D.S.S – 2017) *Let q be a prime and A an elementary abelian q-group of order at least q^2. Suppose that A acts coprimely on a profinite group G and assume that all elements in $C_G(a)$ are Engel in G for each $a \in A^{\#}$. Then G is locally nilpotent.*

There is an example of a finite non-nilpotent group G admitting a four-group of automorphisms A such that $C_G(a)$ is abelian for each $a \in A^{\#}$ can be found for instance in [3]. On the other hand, in [4], we have the following result.

Theorem 7 (C. Acciarri, P. Shumyatsky, D.S.S – 2017) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^{\#}$ every element of $C_G(a)$ is n-Engel in $C_G(a)$. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.*

The statement of a profinite (non-quantitative) version of the above theorem, previously obtained in [3], is as follows.

Theorem 8 (C. Acciarri, P. Shumyatsky – 2016) *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then G is locally nilpotent.*

Let us denote by $\gamma_i(H)$ the ith term of the lower central series of a group H and by $H^{(i)}$ the ith term of the derived series of H. It was shown in [6] and further in [1, 2] that if the rank of the acting group A is big enough, then results of similar nature to that of Theorem 3 can be obtained while imposing conditions on elements of $\gamma_i(C_G(a))$ or $C_G(a)^{(i)}$ rather than on elements of $C_G(a)$. In the same spirit, we extended Theorems 5 and 7 respectively, as follows. The following results were proved in [5].

Theorem 9 (C. Acciarri, D.S.S – 2018) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G.*
(1) If all elements in $\gamma_{r-1}(C_G(a))$ are n-Engel in G for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k.
(2) If, for some integer d such that $2^d \leq r-1$, all elements in the dth derived group of $C_G(a)$ are n-Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k.

Theorem 10 (C. Acciarri, D.S.S – 2018) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 3$ acting on a finite q'-group G.*
(1) If all elements in $\gamma_{r-2}(C_G(a))$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then $\gamma_{r-2}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k.
(2) If, for some integer d such that $2^d \leq r-2$, all elements in the dth derived group of $C_G(a)$ are n-Engel in $C_G(a)$ for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k.

Finally, we formulate the (non-quantitative) analogues of Theorems 9 and 10, respectively.
For a profinite group G, as usual, $\gamma_k(G)$ stands for the smallest normal subgroup closed N of G such that G/N is nilpotent of class at most $k-1$, and $G^{(k)}$ for the smallest normal subgroup closed N of G such that G/N is soluble with derived length at most $k-1$.

Theorem 11 (C. Acciarri, D.S.S – 2018) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting coprimely on a profinite group G.*
(1) If all elements in $\gamma_{r-1}(C_G(a))$ are Engel in G for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ is locally nilpotent.
(2) If, for some integer d such that $2^d \leq r-1$, all elements in the dth derived group of $C_G(a)$ are Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is locally nilpotent.

Theorem 12 (C. Acciarri, D.S.S – 2018) *Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 3$ acting coprimely on a profinite group G.*
(1) If all elements in $\gamma_{r-2}(C_G(a))$ are Engel in $C_G(a)$ for any $a \in A^{\#}$, then $\gamma_{r-2}(G)$ is locally nilpotent.
(2) If, for some integer d such that $2^d \leq r-2$, all elements in the dth derived group of $C_G(a)$ are Engel in $C_G(a)$ for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is locally nilpotent.

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